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$N = 1$ Supersymmetric Extension of the QCD Effective Action

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ABSTRACT

We present a new 4D, $N = 1$ supersymmetric nonlinear σ -model using complex linear and chiral superfields that generalizes the massless limit of the QCD effective action of Gasser and Leutwyler.

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I. Introduction

It is well known that the QCD low-energy effective action is accurately described by a (broken) $SU(3)_L \otimes SU(3)_R$ chiral model encoding the interactions of the lightest flavor- $SU(3)$ meson octet [1]. A systematic expansion in momenta has been given by Gasser and Leutwyler [2]. Though we know that supersymmetry is not a feature of low-energy phenomenology, a great deal of qualitative information about field theories in general has been learned from the study of supersymmetric theories and it is of interest to study the low-energy effective actions for $N = 1$ supersymmetric QCD.

Earlier studies have been formulated solely in terms of chiral superfields [3]. Recently, one of us [4] has proposed a new class of models to describe supersymmetric extensions of the low-energy QCD effective action. These models use both chiral (C) and nonminimal (N), complex linear, superfields to describe scalar multiplets, and split the physical component fields between them. We refer to them as CNM models. The splitting is essentially heterodexterous with “right-handed” matter contained in C-superfields and “left-handed” matter contained in N-superfields. The spinor superpartners of the pions in the CNM models are most naturally described by Dirac fields. Geometrically, the chiral superfields are coordinates in the cotangent bundle of the σ -model manifold considered as a base manifold, while the complex linear superfields coordinatize the fibers.

It is always possible, using suitable duality transformations, to *formally* replace the nonminimal superfields by chiral superfields. To leading order in (spinor) derivatives the resulting theory looks like the proposal of Rohm and Nemeschansky [3]. It differs by higher derivative terms and by having twice as many fields, which enables the spinor superpartners of the pions to also be Dirac fields in the dual version of the CNM model. Furthermore, the CNM formulation allows an *explicit* mapping from the bosonic effective action to the $N = 1$ supersymmetric effective action, and exhibits some interesting geometrical features [4].

In the simplest case, we are interested in constructing a 4D, $N = 1$ superfield action with the property that if only the pion octet is retained, the manifestly supersymmetric action should come as close as possible to being in agreement with the Gasser-Leutwyler parametrization. An important ingredient in this is the explicit realization of chiral $SU(3)_L \otimes SU(3)_R$ symmetry in terms of superfields. To simplify matters further, we work in the limit where all masses vanish and all spin-1 fields are set to zero. What emerges is a simple form that is the topic of this Letter.

II. The Massless Limit of the Gasser-Leutwyler Action

We recall that the pion octet can be introduced as an element of the $SU(3)$ algebra in the form,

$$\frac{1}{f_\pi} \Pi \equiv \frac{1}{f_\pi} \Pi^i t_i = \frac{1}{f_\pi} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\eta\sqrt{\frac{2}{3}} \end{pmatrix} . \quad (1)$$

Here t_1, \dots, t_8 are the Gell-Mann $SU(3)$ matrices and f_π is the pion decay constant. Exponentiation of the Lie algebraic element in (1) via the definition $U \equiv \exp[i\frac{1}{f_\pi}\Pi^i t_i]$ then leads to elements in the $SU(3)$ group. Other useful quantities are the Maurer-Cartan forms R_m^i and L_m^i introduced via

$$U^{-1} \partial_{\underline{a}} U = i f_\pi^{-1} (\partial_{\underline{a}} \Pi^m) R_m^i(\Pi) t_i \quad , \quad (\partial_{\underline{a}} U) U^{-1} = i f_\pi^{-1} (\partial_{\underline{a}} \Pi^m) L_m^i(\Pi) t_i . \quad (2)$$

(We are using underlined letters to denote space-time vector indices, for later use in the supersymmetric case.)

These definitions allow $R_m^i(\Pi)$ and $L_m^i(\Pi)$ to be calculated as power series in Π^i from

$$\begin{aligned} R_m^i(\Pi) &\equiv (C_2)^{-1} \text{Tr} \left[t^i \left(\frac{1 - e^{-\Delta}}{\Delta} \right) t_m \right] , \\ L_m^i(\Pi) &\equiv (C_2)^{-1} \text{Tr} \left[t^i \left(\frac{e^{\Delta} - 1}{\Delta} \right) t_m \right] , \end{aligned} \quad (3)$$

where $\Delta t_m \equiv i f_\pi^{-1} [\Pi, t_m]$, $\Delta^2 t_m = \Delta \Delta t_m$, etc. and the constant C_2 is determined so that $L_m^i(0) = R_m^i(0) = \delta_m^i$.

As a consequence, the actions (which occur in [2])

$$\mathcal{S}_\sigma = \frac{f_\pi^2}{2C_2} \int d^4x \text{Tr}[(\partial^{\underline{a}} U^{-1}) (\partial_{\underline{a}} U)] , \quad (4)$$

$$\mathcal{S}_1 = L_1 \int d^4x \left(\text{Tr}[(\partial^{\underline{a}} U^{-1}) (\partial_{\underline{a}} U)] \right)^2 , \quad (5)$$

$$\mathcal{S}_2 = L_2 \int d^4x \text{Tr}[(\partial_{\underline{a}} U^{-1}) (\partial_{\underline{b}} U)] \text{Tr}[(\partial^{\underline{a}} U^{-1}) (\partial^{\underline{b}} U)] , \quad (6)$$

$$\mathcal{S}_3 = L_3 \int d^4x \text{Tr}[(\partial_{\underline{a}} U^{-1}) (\partial_{\underline{b}} U) (\partial^{\underline{a}} U^{-1}) (\partial^{\underline{b}} U)] , \quad (7)$$

can be re-expressed in terms of the Maurer-Cartan forms. In fact there is some redundancy in (5) and (6), and also (7) is not written in a manner that is most

convenient to discuss the possible supersymmetric extension. To sort this out, it is convenient to introduce irreducible projection operators,

$$\begin{aligned} P^{(0)} \underline{a} \underline{b} \underline{c} \underline{d} &\equiv \frac{1}{4} \eta^{\underline{a} \underline{b}} \eta^{\underline{c} \underline{d}} \quad , \quad P^{(1)} \underline{a} \underline{b} \underline{c} \underline{d} \equiv \frac{1}{2} \left[\eta^{\underline{a} \underline{c}} \eta^{\underline{d} \underline{b}} - \eta^{\underline{a} \underline{d}} \eta^{\underline{c} \underline{b}} \right] \quad , \\ P^{(2)} \underline{a} \underline{b} \underline{c} \underline{d} &\equiv \frac{1}{2} \left[\eta^{\underline{a} \underline{c}} \eta^{\underline{d} \underline{b}} + \eta^{\underline{a} \underline{d}} \eta^{\underline{c} \underline{b}} - \frac{1}{2} \eta^{\underline{a} \underline{b}} \eta^{\underline{c} \underline{d}} \right] \quad . \end{aligned} \quad (8)$$

It follows from the completeness of these operators that we may write

$$\mathcal{S}_2 = \mathcal{S}_2^{(0)} + \mathcal{S}_2^{(1)} + \mathcal{S}_2^{(2)} \quad , \quad (9)$$

$$\mathcal{S}_3 = \mathcal{S}_3^{(0)} + \mathcal{S}_3^{(1)} + \mathcal{S}_3^{(2)} \quad , \quad (10)$$

where

$$\mathcal{S}_2^{(i)} \equiv L_2^{(i)} \int d^4 x P^{(i)} \underline{a} \underline{b} \underline{c} \underline{d} \text{Tr}[(\partial_{\underline{a}} U^{-1})(\partial_{\underline{b}} U)] \text{Tr}[(\partial_{\underline{c}} U^{-1})(\partial_{\underline{d}} U)] \quad , \quad (11)$$

$$\mathcal{S}_3^{(i)} \equiv L_3^{(i)} \int d^4 x P^{(i)} \underline{a} \underline{b} \underline{c} \underline{d} \text{Tr}[(\partial_{\underline{a}} U^{-1})(\partial_{\underline{b}} U)(\partial_{\underline{c}} U^{-1})(\partial_{\underline{d}} U)] \quad . \quad (12)$$

Clearly \mathcal{S}_1 and $\mathcal{S}_2^{(0)}$ have the same functional form so that one of them is redundant. It is therefore consistent to set \mathcal{S}_1 to zero and use $\mathcal{S}_2^{(0)}$ to parametrize its contribution to physical processes. In the work by Gasser and Leutwyler, *a priori* assumptions that $L_2^{(0)} = L_1 + L_2$, $L_2^{(1)} = L_2$, $L_2^{(2)} = L_2$ and $L_3^{(i)} = L_3$ were made. Finally, we note that $\mathcal{S}_3^{(1)}$ describes the familiar ‘‘Skyrme’’ term [5], whose supersymmetric generalization was attempted in ref. [6].

Thus to this order, the low-energy QCD effective action may be parametrized in the form

$$\mathcal{S}_{eff}(QCD) = \mathcal{S}_\sigma + \sum_{i=0}^2 \left[\mathcal{S}_2^{(i)} + \mathcal{S}_3^{(i)} \right] + \mathcal{S}_{WZNW} \quad , \quad (13)$$

where we have dropped the assumptions mentioned above and have added in the Wess-Zumino-Novikov-Witten term as well. As shown by Witten [7], it is most convenient to define an extended group element \widehat{U} . Thus we define $\widehat{U} \equiv \exp[iy f_\pi^{-1} \Pi]$ and in terms of the extended group element, the WZNW term is given by

$$\begin{aligned} \mathcal{S}_{WZNW} &= -iN_C [2 \cdot 5!]^{-1} \int d^4 x \int_0^1 dy \text{Tr} \left[(\widehat{U}^{-1} \partial_y \widehat{U}) \widehat{\mathcal{W}}_4 \right] \quad , \\ \widehat{\mathcal{W}}_4 &= \epsilon^{\underline{a} \underline{b} \underline{c} \underline{d}} (\partial_{\underline{a}} \widehat{U}^{-1})(\partial_{\underline{b}} \widehat{U})(\partial_{\underline{c}} \widehat{U}^{-1})(\partial_{\underline{d}} \widehat{U}) \quad . \end{aligned} \quad (14)$$

We may rewrite the higher derivative terms in (13) in terms of space-time derivatives of the pion fields, using

$$\begin{aligned} \partial_{\underline{a}} U &= \left(\frac{\partial U}{\partial \Pi^i} \right) (\partial_{\underline{a}} \Pi^i) \equiv (\partial_i U) (\partial_{\underline{a}} \Pi^i) \quad , \\ (\partial_{\underline{a}} U^{-1})(\partial_{\underline{b}} U) &= (\partial_i U^{-1})(\partial_j U) (\partial_{\underline{a}} \Pi^i)(\partial_{\underline{b}} \Pi^j) \equiv \mathcal{Z}_{ij} (\partial_{\underline{a}} \Pi^i)(\partial_{\underline{b}} \Pi^j) \quad . \end{aligned} \quad (15)$$

Here we see the appearance of factors of the form $\partial_{\underline{a}}\Pi^i$, “the pullback” from the group manifold to the spacetime manifold.

We note in (14) that the pullbacks with our definitions are y -independent, thus:

$$\mathcal{S}_{WZNW} = -iN_C [2\cdot 5!]^{-1} \int d^4x \epsilon^{\underline{a}_1 \underline{a}_2 \underline{a}_3 \underline{a}_4} \beta_{i_1 i_2 i_3 i_4}(\Pi) \left(\prod_{j=1}^4 \partial_{\underline{a}_j} \Pi^{i_j} \right) , \quad (16)$$

$$\beta_{ijkl}(\Pi) \equiv \int_0^1 dy \text{Tr} \left[(\hat{U}^{-1} \partial_y \hat{U}) (\partial_i \hat{U}^{-1}) (\partial_j \hat{U}) (\partial_k \hat{U}^{-1}) (\partial_l \hat{U}) \right] .$$

It is clear that there exist polynomials $\mathcal{J}_{i_1 i_2 i_3 i_4}^{A2}$ and $\mathcal{J}_{i_1 i_2 i_3 i_4}^{A3}$ which are quadratic in \mathcal{Z}_{ij} such that

$$\mathcal{S}_2 = \int d^4x \sum_{A=0}^2 L_2^{(A)} P^{(A)}{}_{\underline{a}_1 \dots \underline{a}_4} \text{Tr} \left[\mathcal{J}_{i_1 i_2 i_3 i_4}^{A2}(\mathcal{Z}) \right] \left(\prod_{j=1}^4 \partial_{\underline{a}_j} \Pi^{i_j} \right) , \quad (17)$$

$$\mathcal{S}_3 = \int d^4x \sum_{A=0}^2 L_3^{(A)} P^{(A)}{}_{\underline{a}_1 \dots \underline{a}_4} \text{Tr} \left[\mathcal{J}_{i_1 i_2 i_3 i_4}^{A3}(\mathcal{Z}) \right] \left(\prod_{j=1}^4 \partial_{\underline{a}_j} \Pi^{i_j} \right) , \quad (18)$$

where $P^{(0)}{}_{\underline{a}_1 \dots \underline{a}_4}$, $P^{(1)}{}_{\underline{a}_1 \dots \underline{a}_4}$ and $P^{(2)}{}_{\underline{a}_1 \dots \underline{a}_4}$ were defined in (8). We note that higher order (in momentum) terms of the QCD effective action have the same general form.

III. The CNM Model on the Group Manifold

It is our goal to describe an $N = 1$ supersymmetric theory that contains an appropriate generalization of each of the terms that appear in (13). In ref. [4], a new proposal was made as to how manifestly 4D, $N = 1$ supersymmetric formulations containing (16), (17) and (18) might be obtained. The key idea is that all the spin-0 and spin-1/2 fields do *not* occur as the components of chiral scalar superfields (i.e. Wess-Zumino multiplets) as in the standard orthodoxy [8] for such constructions. Instead it was proposed that the right-handed components of Dirac fields be embedded into Wess-Zumino chiral scalar superfields Φ , $\bar{D}_{\dot{\alpha}}\Phi = 0$ and the left-handed components of Dirac fields be embedded into “non-minimal” scalar multiplets [9] (i.e. complex linear superfields) Σ , $\bar{D}^2\Sigma = 0$. Supersymmetrizing the nonlinear σ -model necessarily leads to a “parity-doubling” of the basic degrees of freedom: supersymmetry requires a scalar partner for every pseudoscalar. Here, we double once more by introducing the complex linear fields Σ .

The main benefit of our approach is twofold: (a) the equations of motion of auxiliary fields are *always* purely algebraic so that the auxiliary fields, which do not propagate at the free lagrangian level, remain non-propagating – we refer to this

feature as *auxiliary freedom*; and (b.) there is an obvious holomorphy condition that we can impose. (In the final section we will find ways of rewriting our action that makes this holomorphy condition less natural. However, dropping holomorphy *may* lead to propagation of auxiliary fields.)

Let Φ^I be a set of 4D, $N = 1$ chiral superfields. Define chiral superfield group elements by

$$U(\Phi) \equiv \exp \left[\frac{\Phi^I t_I}{f_\pi \cos(\gamma_S)} \right] \quad , \quad (19)$$

where t_I denote the hermitian matrix generators of some compact Lie algebra. Here γ_S is a mixing angle (see below) which is restricted by the form of the supersymmetric WZNW action to satisfy the condition $\sin(2\gamma_S) \neq 0$ [4]. Due to the complex nature of chiral superfields, we are working in the complexification of the group with generators t_I , and

$$U^\dagger \neq U^{-1} \quad . \quad (20)$$

More explicitly we may write

$$U(\Phi) = \exp \left[\frac{i}{f_\pi \cos(\gamma_S)} \left(-\text{Re}(\Phi^I)(it_I) + \text{Im}(\Phi^I)(t_I) \right) \right] \quad , \quad (21)$$

and thus the generators of the group are $\{t_I, it_I\}$.

Right chiral superfield Maurer-Cartan forms $R_I^K(\Phi)$ and left chiral superfield Maurer-Cartan forms $L_I^K(\Phi)$ are defined by

$$\begin{aligned} U^{-1} D_\alpha U &= [f_\pi \cos(\gamma_S)]^{-1} (D_\alpha \Phi^I) R_I^K(\Phi) t_K \quad , \\ (D_\alpha U) U^{-1} &= [f_\pi \cos(\gamma_S)]^{-1} (D_\alpha \Phi^I) L_I^K(\Phi) t_K \quad , \end{aligned} \quad (22)$$

and $R_I^K(\Phi)$ and $L_I^K(\Phi)$ can be calculated as in (3). We denote the matrix inverses of $R_I^K(\Phi)$ and $L_I^K(\Phi)$ by $(R^{-1})_K^I$ and $(L^{-1})_K^I$, respectively. Since the multiplication of chiral superfields is closed we also observe,

$$\begin{aligned} U^{-1} \overline{D}_{\dot{\alpha}} U &= 0 \quad , \quad U \overline{D}_{\dot{\alpha}} U^{-1} = 0 \quad \rightarrow \quad \overline{D}_{\dot{\alpha}} R_I^K = \overline{D}_{\dot{\alpha}} L_I^K = 0 \quad , \\ \overline{D}_{\dot{\alpha}} \left[U^{-1} D_\alpha U \right] &= i [f_\pi \cos(\gamma_S)]^{-1} (\partial_{\underline{a}} \Phi^I) R_I^K(\Phi) t_K \quad , \\ \overline{D}_{\dot{\alpha}} \left[(D_\alpha U) U^{-1} \right] &= i [f_\pi \cos(\gamma_S)]^{-1} (\partial_{\underline{a}} \Phi^I) L_I^K(\Phi) t_K \quad . \end{aligned} \quad (23)$$

We use a spinor notation where vector indices are denoted by $\underline{a} \equiv \alpha \dot{\alpha}$.

We consider the rigid $SU_L(3) \otimes SU_R(3)$ transformations defined by

$$(U)' = \exp[-i\tilde{\alpha}^I t_I] U \exp[i\alpha^I t_I] \quad , \quad (24)$$

with left and right transformation parameters $\tilde{\alpha}^I$ and α^I . Note that these are *not* complexified transformations (i.e. $\tilde{\alpha}^I$ and α^I are real). Infinitesimally, using Maurer-Cartan forms, this can be written as a variation of the chiral superfields,

$$\delta\Phi^I = \alpha^{(A)}\xi_{(A)}^I(\Phi) \quad , \quad (25)$$

where

$$\alpha^{(A)}\xi_{(A)}^I \equiv -i[f_\pi \cos(\gamma_S)] [\tilde{\alpha}^J(L^{-1})_J^I - \alpha^J(R^{-1})_J^I] \quad , \quad (26)$$

or, in finite form, as a coordinate transformation,

$$(\Phi^I)' = K^I(\Phi) = \exp[\alpha^{(A)}\xi_{(A)}^J \partial_J] \Phi^I \quad , \quad \partial_J \equiv \partial/\partial\Phi^J \quad . \quad (27)$$

As described in ref. [4], we also introduce nonminimal scalar multiplets described by complex linear superfields Σ^I , and we split the physical fields between the components of Φ and Σ . In particular, for the bosonic fields, we write (as usual, the vertical bar indicates evaluation at $\theta = 0$)

$$\begin{aligned} \Phi^I| &= A^I(x) = \mathcal{A}^I(x) + i \left[\Pi^I(x) \cos(\gamma_S) + \Theta^I(x) \sin(\gamma_S) \right] \quad , \\ \Sigma^I| &= B^I(x) = \mathcal{B}^I(x) + i \left[-\Pi^I(x) \sin(\gamma_S) + \Theta^I(x) \cos(\gamma_S) \right] \quad , \end{aligned} \quad (28)$$

in terms of two real octets of scalar spin-0 fields \mathcal{A}^I and \mathcal{B}^I as well as two real octets of pseudo-scalar spin-0 fields Π^I and Θ^I . As was discussed in ref. [4], the consistency of the model requires $\sin(2\gamma_S) \neq 0$.

We postulate that the nonminimal multiplets transform under the transformations (24) as 1-forms (or cotangent vectors)

$$(\Sigma^L)' = \left(\partial_I K^L \right) \Sigma^I \quad , \quad (29)$$

We may, however, convert them into fields that transform as the group elements U by introducing matrix valued fields $\hat{\Sigma}$:

$$\hat{\Sigma} \equiv \left(\partial_I U \right) \Sigma^I \quad . \quad (30)$$

We emphasize that $(\Sigma^L)'$ and $\hat{\Sigma}$ remain complex linear because K and $\partial_I U$ are chiral and the product of a chiral and a linear superfield is linear.

IV. 4D, N = 1 CNM Supersymmetric QCD Effective Action

Any function of traces of *appropriate* products of U , U^\dagger , $\widehat{\Sigma}$, $\widehat{\Sigma}^\dagger$ is automatically invariant under rigid $SU_L(3) \otimes SU_R(3)$ transformations. The minimal choice for the σ -model action is (where \mathcal{N}_0 , \mathcal{N}_1 are normalization constants)

$$\mathcal{S}_\sigma(\Phi, \Sigma) = \int d^4x d^2\theta d^2\bar{\theta} \left[f_\pi^2 \mathcal{N}_0 \text{Tr}[U^\dagger U] - \mathcal{N}_1 \text{Tr}[\widehat{\Sigma}^\dagger \widehat{\Sigma}] \right] . \quad (31)$$

We propose this action as the CNM chiral model which for the group $SU(3)$ corresponds to the leading term of the 4D, N = 1 QCD effective action. Note that at this point, $\widehat{\Sigma}$ is a decoupled free field.

Having achieved this reformulation of previous results [4] in such a way that the rigid $SU_L(3) \otimes SU_R(3)$ symmetry is manifest in the superfield action, it is a simple step to reformulate the Skyrme and WZNW terms described previously [4]. Consider the real parts of the following superfield actions

$$\begin{aligned} \mathcal{S}_{\text{Skyrme}}(\Phi, \Sigma) &= C_1 \int d^4x d^2\theta \eta^{a[c} \eta^{d]b} \text{Tr} \left[(\partial_I U^{-1})(\partial_J U)(\partial_K U^{-1})(\partial_L U) \right] \\ &\quad \times (\partial_{\underline{a}} \Phi^I)(\partial_{\underline{b}} \Phi^J) C_{\gamma\delta} (\overline{D}_{\dot{\gamma}} \Sigma^K)(\overline{D}_{\dot{\delta}} \Sigma^L) , \\ \mathcal{S}_{\text{WZNW}}(\Phi, \Sigma) &= i C_0 \int d^4x d^2\theta \epsilon^{abcd} \mathcal{J}_{IJKL} (\partial_{\underline{a}} \Phi^I)(\partial_{\underline{b}} \Phi^J) C_{\gamma\delta} (\overline{D}_{\dot{\gamma}} \Sigma^K)(\overline{D}_{\dot{\delta}} \Sigma^L) , \end{aligned} \quad (32)$$

where \mathcal{J}_{IJKL} is simply β_{ijkl} of equation (15) evaluated for the chiral superfield U . These actions are completely equivalent to the explicit forms that were given in reference [4]. Written in this way, it is manifest that these higher derivative terms are also invariant under the $SU_L(3) \otimes SU_R(3)$ rigid transformations. We note that the Skyrme term is free of some of the problems that plagued, as recognized by its authors, the proposal in ref. [6] (however, just as our model, the proposal in [3] does not suffer from propagating auxiliary fields).

To define the most general members of the CNM group manifold models with rigid $SU_L(3) \otimes SU_R(3)$ symmetry, we add to (31) the integral over full superspace of the real part of any function of traces of appropriate products of U , U^\dagger , $\widehat{\Sigma}$, $\widehat{\Sigma}^\dagger$ as well as the real part of the chiral integral of any function of traces of U and $\overline{D}\widehat{\Sigma}$ (with \overline{D} 's contracted pairwise). It can be verified that all of these actions are at most quadratic in spacetime derivatives of bosonic fields. Thus, these may be regarded as deformations of the basic σ -model action in (31).

To generalize to higher derivative terms with rigid $SU_L(3) \otimes SU_R(3)$ symmetry we add to (32) the real part of a chiral integral of a function of traces of U , $\overline{D}\widehat{\Sigma}$, and

$\partial_{\underline{a}}U$ with all possible Lorentz contractions on the spinor indices. Some typical terms might be:

$$\mathcal{S}_{\text{H.D.}}(\Phi, \Sigma) = \int d^4x d^2\theta \sum_{A,p,q,k} L_k^A P_{\dot{\alpha}_1 \dot{\beta}_1 \dots \dot{\alpha}_p \dot{\beta}_p}^{A \underline{c}_1 \dots \underline{c}_q} \times \text{Tr} \left[\mathcal{J}^k(U) \prod_i^p (\overline{D}^{\dot{\alpha}_i} \hat{\Sigma}) (\overline{D}^{\dot{\beta}_i} \hat{\Sigma}) \left(\prod_j^q \partial_{\underline{c}_j} U \right) \right] , \quad (33)$$

where $P_{\dot{\alpha}_1 \dot{\beta}_1 \dots \dot{\alpha}_p \dot{\beta}_p}^{A \underline{c}_1 \dots \underline{c}_q}$ denote the distinct Lorentz invariant tensors that may be written for a fixed number of indices. For example, in the case of the Skyrme and WZNW terms we found

$$P_{\text{Skyrme}}^{\underline{c} \underline{d} \dot{\alpha} \dot{\beta}} = \eta^{\underline{a} \underline{c}} \eta^{\underline{d} \underline{b}} C_{\alpha \beta} \quad , \quad P_{\text{WZNW}}^{\underline{c} \underline{d} \dot{\alpha} \dot{\beta}} = \epsilon^{\underline{abcd}} C_{\alpha \beta} \quad . \quad (34)$$

In (33) if p is greater than one, then the superfield action when evaluated in terms of component fields will have the property that it possesses no purely bosonic terms. So in a sense the most interesting actions described by (33) are those for which $p = 1$ and $q > 1$. In particular, the simplest of these, with $p = 1$, $q = 2$, leads to a component action containing the higher-derivative pion term [4]

$$\begin{aligned} & \int d^4x d^2\theta \mathcal{J}_{\text{IJKL}}(\Phi) (\overline{D}^{\dot{\alpha}} \Sigma^{\text{I}}) (\overline{D}^{\dot{\beta}} \Sigma^{\text{J}}) (\partial^{\gamma}_{\dot{\alpha}} \Phi^{\text{K}}) (\partial_{\gamma \dot{\beta}} \Phi^{\text{L}})|_{\text{pion}} \\ &= \frac{1}{4} \sin^2(2\gamma_{\text{S}}) \int d^4x \mathcal{J}_{\text{IJKL}}(\Pi) P^{\underline{a} \underline{b} \underline{c} \underline{d}} \left[(\partial_{\underline{a}} \Pi^{\text{I}}) (\partial_{\underline{b}} \Pi^{\text{J}}) (\partial_{\underline{c}} \Pi^{\text{K}}) (\partial_{\underline{d}} \Pi^{\text{L}}) \right] . \end{aligned} \quad (35)$$

We emphasize that all of these actions are *auxiliary-free*: at the component level the auxiliary fields *do not* propagate in spite of the higher-derivative terms.

Although our formulation in terms of chiral $d^2\theta$ integrals, based on the simple but key observation that $\overline{D}^{\dot{\alpha}} \Sigma$ is chiral [4], is consistent, we now know that it is not unique. In fact, terms of the type in (33), and specifically in (35), *can* be rewritten as full superspace integrals:

$$\begin{aligned} & \int d^4x d^2\theta \mathcal{J}_{\text{IJKL}}(\Phi) (\overline{D}^{\dot{\alpha}} \Sigma^{\text{I}}) (\overline{D}^{\dot{\beta}} \Sigma^{\text{J}}) (\partial^{\gamma}_{\dot{\alpha}} \Phi^{\text{K}}) (\partial_{\gamma \dot{\beta}} \Phi^{\text{L}}) \\ &= -i \int d^4x d^4\theta \mathcal{J}_{\text{IJKL}}(\Phi) \Sigma^{\text{I}} (\overline{D}^{\dot{\beta}} \Sigma^{\text{J}}) (D^{\gamma} \Phi^{\text{K}}) (\partial_{\gamma \dot{\beta}} \Phi^{\text{L}}) . \end{aligned} \quad (36)$$

Having rewritten this term in the action in this form, there is no longer any obvious reason to restrict \mathcal{J} to be a function of chiral superfields only; indeed, group invariance is maintained even when \mathcal{J} is a function not only of Φ , but of $\overline{\Phi}$, Σ , $\overline{\Sigma}$ as well.

One of the issues that was raised (and not answered) in the second work of ref. [4] was whether it is possible that some auxiliary-free higher derivative terms can

arise from integrals over the full superspace instead of chiral superspace integrals as in (33). The answer to this is yes. It is a straight forward calculation to prove that the following expression (which is *not* in general a rewriting of a chiral integral)

$$\begin{aligned} \mathcal{S}'_{\text{H.D.}}(\Phi, \Sigma) = & \int d^4x d^2\theta d^2\bar{\theta} \sum_{A,k,p,q,r,s} L'_k{}^A P'^A_{\{\alpha_i\}\{\dot{\alpha}_i\}} \text{Tr} \left[\mathcal{J}_{\{I_i\}\{J_i\}}^{A k \{K_i\} \{L_i\}}(U) \right] \\ & \times \left(\prod_{i=1}^p \Sigma^{I_i} \right) \left(\prod_{i=1}^q \bar{D}_{\dot{\alpha}_i} \Sigma^{J_i} \right) \left(\prod_{i=1}^r D_{\alpha_i} \Phi^{K_i} \right) \left(\prod_{i=1}^s \partial_{\underline{a}_i} \Phi^{L_i} \right) \quad , \end{aligned} \quad (37)$$

leads to an auxiliary free higher derivative action. Thus, the analog of (13) takes the form

$$\mathcal{S}_{eff}^{SUSY}(QCD) = \mathcal{S}_{\sigma}(\Phi, \Sigma) + \mathcal{S}'_{\text{H.D.}}(\Phi, \Sigma) + \mathcal{S}_{WZ\text{NW}}(\Phi, \Sigma) \quad , \quad (38)$$

where the P -symbols in (33) can be related to those in (13) by

$$P^A{}_{\underline{\varepsilon}_1 \underline{\varepsilon}_2 \dot{\alpha} \dot{\beta}} = P^{(A)}{}_{\underline{\varepsilon}_1, \beta \dot{\gamma}_2, \underline{a}, \gamma_2 \dot{\beta}} C_{\alpha \beta} \quad (39)$$

(with, as usual, $\underline{b} \equiv \beta \dot{\beta}$ etc). To the same order as the Gasser-Leutwyler result we would only let $q = 2$. In general, however, the effective action of (29) will include terms with q an arbitrary even integer.

V. The CNM Map

Having achieved our goal of providing a supersymmetric generalization of (13), it is apparent that more has also been accomplished. To all orders, the derivative terms of the pion sector of QCD effective action is expected to be of the form

$$\mathcal{S}_{eff}(QCD) = \mathcal{S}_{\sigma} + \sum_{i=0}^{\infty} \left[\mathcal{S}^{(i)} \right] + \mathcal{S}_{WZ\text{NW}} \quad , \quad (40)$$

$$\mathcal{S}^{(i)} = \int d^4x \sum_{A,B,k} L_k^{(A)(B)} P^{(A)}{}_{\underline{a}_1 \dots \underline{a}_{2i}} P^{(B)}{}_{\dot{a}_1 \dots \dot{a}_{2i}} \text{Tr} \left[\mathcal{J}_{k_1 \dots k_{2i}}^{A B k}(\mathcal{Z}) \right] \left(\prod_{j=1}^{2i} \partial_{\underline{a}_j} \Pi^{i_j} \right) \quad , \quad (41)$$

where for completeness, we have introduced a set of irreducible projection operators that act on the i -indices. We can define a mapping \mathcal{G}_S^C on higher derivative bosonic terms to the supersymmetric ones via the following simple rules,

$$\begin{aligned} \mathcal{G}_S^C : \exp[i \frac{1}{f_{\pi}} \Pi^i t_i] & \rightarrow \exp \left[\frac{\Phi^I t_I}{f_{\pi} \cos(\gamma_S)} \right] \quad , \\ \mathcal{G}_S^C : \int d^4x & \rightarrow \int d^4x \int d^2\theta \quad , \\ \mathcal{G}_S^C : \left(\prod_{j=1}^q \partial_{\underline{\varepsilon}_j} \Pi^{i_j} \right) & \rightarrow C_{\gamma_1 \gamma_q}(\bar{D}_{\dot{\gamma}_1} \Sigma^{I_1}) (\bar{D}_{\dot{\gamma}_q} \Sigma^{I_q}) \left(\prod_{j=2}^{q-1} \partial_{\underline{\varepsilon}_j} \Phi^{I_j} \right) \quad , \end{aligned} \quad (42)$$

for chiral actions. Similarly non-chiral higher derivative actions are obtained by using \mathcal{G}_S where

$$\begin{aligned}\mathcal{G}_S : \exp[i\frac{1}{f_\pi}\Pi^i t_i] &\rightarrow \exp\left[\frac{\Phi^I t_I}{f_\pi \cos(\gamma_S)}\right] , \\ \mathcal{G}_S : \int d^4x &\rightarrow \int d^4x \int d^2\theta \int d^2\bar{\theta} , \\ \mathcal{G}_S : \left(\prod_{j=1}^q \partial_{\underline{c}_j} \Pi^{i_j}\right) &\rightarrow \left(\prod_{i=1}^P \Sigma^{I_i}\right) \left(\prod_{i=1}^Q \bar{D}_{\dot{\alpha}_i} \Sigma^{J_i}\right) \left(\prod_{i=1}^R D_{\alpha_i} \Phi^{K_i}\right) \left(\prod_{i=1}^S \partial_{\underline{c}_i} \Phi^{L_i}\right) .\end{aligned}\tag{43}$$

where $q = P + Q + R + S$. Thus, the most striking feature of the CNM approach is that it easily permits *exactly the same polynomials \mathcal{J}^{ABk} that determine the higher derivative terms of the non-supersymmetric QCD effective action to also determine the higher derivative terms of a 4D, $N = 1$ supersymmetric QCD effective action.*

VI. Duality Transformation

We now focus on the action (31) with the added term (35) rewritten as a full superspace integral as in (36), and perform a duality transformation to a theory with chiral and anti-chiral superfields only [9]. As usual, this is done by relaxing the constraint on the superfield to be dualized, in this case $\widehat{\Sigma}$, and imposing it by adding a Lagrange multiplier field, in this case a chiral superfield χ . Integrating out the Lagrange multiplier reimposes the constraint and gives back the model in CNM language, whereas integrating out the unconstrained field gives the dual model. Explicitly, we replace (31) and (36) by

$$\begin{aligned}\mathcal{S}_\sigma(\Phi, X) = \int d^4x d^2\theta d^2\bar{\theta} &\left(Tr[U^\dagger U] - Tr[\widehat{X}^\dagger \widehat{X} - \chi \widehat{X} - \widehat{X}^\dagger \chi^\dagger] \right. \\ &\left. - \left[i \mathcal{J}_{IJKL}(\Phi) X^I (\bar{D}^{\dot{J}} X^J) (D^\gamma \Phi^K) (\partial_{\gamma\dot{\beta}} \Phi^L) + \text{h.c.} \right] \right) ,\end{aligned}\tag{44}$$

where \widehat{X} is an unconstrained field replacing $\widehat{\Sigma}$. Integrating out \widehat{X} and its conjugate gives an equation that can be solved iteratively for them in terms of Φ, χ , their conjugates, and their derivatives. We find

$$\widehat{X} = \chi^\dagger + \dots\tag{45}$$

where the remaining terms are higher order in spinor and vector derivatives. Substituting back, to leading order, we find an action of the form proposed by [3], but with twice as many superfields.

VII. Conclusion

We have presented a superspace formulation of the QCD effective action in terms of chiral and linear superfields, which can be used to map any bosonic action into a corresponding supersymmetric one. Within the purely chiral superfield sector of the theory, since it is a 4D, $N = 1$ supersymmetric non-linear σ -model, the CNM action defined by equation (38) obviously describes a Kähler manifold with potential⁶ $K(\Phi, \bar{\Phi}) = \text{Tr}[U U^\dagger]$. More explicitly, it may be written as a finite sum of products of holomorphic and antiholomorphic functions, $K(\Phi, \bar{\Phi}) = \sum f^I(\Phi) \bar{f}^I(\bar{\Phi})$ for some suitable functions⁷ f^I . For obvious reasons, we may call this a “group” Kähler potential and the geometry associated with this “group Kähler geometry.”

An interesting feature of the CNM formulation of the $N = 1$ supersymmetric low-energy QCD effective action is the ‘prediction’ of a new physical constant, the non-vanishing mixing angle γ_S , with $\sin(2\gamma_S) \neq 0$. In the CNM model of the supersymmetric low-energy QCD effective action, requiring the presence of higher order derivative terms that contain a component field which can be identified as the pion octet (see eq. (41)) imposes this restriction on γ_S . This angle is reminiscent of the “weak mixing angle” θ_W of the Glashow-Salam-Weinberg model of the Electroweak Interaction which is also restricted by theoretical reasons to satisfy $\sin(2\theta_W) \neq 0$ in order that the photon couple to a *purely* vector current.

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⁶To our knowledge, this explicit form of the Kähler potential was first suggested in ref. [10].

⁷These functions correspond to the “old” σ and $\vec{\pi}$ variables in the original works on σ -models.

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